RESEARCH PROPOSAL

A STUDY OF EXTENSION OF GAMMA AND BETA FUNCTIONS AND FRACTIONAL DIFFER-INTEGRAL OPERATOR

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Contents

1 Introduction **2**

2 Review of Literature **2**

3 Motivation **6**
3.1 Some definitions and earlier work 6
3.2 Summary of the proposed work 10

4 Objective **12**

5 Plan of work and Methodology **13**
5.1 Plan of Work 13
5.2 Methodology 13

Bibliography **14**

1 Introduction

Special functions are particular mathematical functions which have more or less established names and notations due to their importance in mathematical analysis,functional analysis, physics or other applications.

Special functions are class of mathematical functions that arise in the solution of various classical problems of physics. These problems generally involve the flow of electromagnetic, acoustic, or thermal energy.

Most of named functions (among them: the Bessel and all cylindrical functions; the Gauss, Confluent and generalized hypergeometric functions; the classical orthogonal polynomials, the incomplete Gamma and Beta functions and Error functions, the Airy, Whittaker etc. functions) provide solutions to integer order differential equations and systems, used as mathematical models.

However, recently there has been an increasing interest in and widely extended use of differential equations and systems of fractional order (that is, of arbitrary order), as better models of phenomena of various physical, engineering, automatization, biological and biomedical, chemical, Earth science, economic,etc.

fFractional differ-integral Operators(or Fractional calculus) is a generalization of ordinary differentiation and integration to arbitrary (non-integer) order. The subject is as old as the differential calculus, and goes back to times when Leibnitz and Newton invented differential calculus.

The subject of fractional calculus and its applications in various and widespread fields of science and engineering have been proposed. A list of the fields of application may include acoustic wave propagation in inhomogeneous porous material, diffusive transport, fluid flow, dynamical processes in self-similar structures, dynamics of earthquakes, optics, geology,viscoelastic materials, bio-sciences, medicine, economics, probability and statistics, astrophysics, chemical engineering,physics, electromagnetic waves, nonlinear control, signal processing,control of power electronicconverters, chaotic dynamics

The "Special Functions of Fractional Calculus" are then the necessary tool, related to the theory of differentiation and integration of arbitrary order and to the fractional order (or multi-order) differential and integral equations, that needs now its unified presentation and extensive development.

2 Review of Literature

In 2004, Chaudhry et al. [7] used extended beta function to extend the hypergeometric functions and confluent hypergeometric functions. They called these functions extended Gauss hypergeometric function (EGHF) and extended confluent hypergeometric function (ECHF), respectively. They obtained the differentiation properties, Mellin transforms of EGHF, transformation formulas, recurrence relations, summation and asymptotic formulas for this function.

In 2011,Ozergin et al. [17] introduced generalizations of the gamma and Euler's beta functions.Through this generalized beta function they introduced generalized Gauss hypergeometric and confluent hypergeometric functions. Some recurrence relations, transformation formulas, operation formulas and integral representations were also obtained for these new generalizations.

In 2014, Choi et al. [10] presented a further extension of the extended beta function and gave connection with other special functions and polynomials such as Macdonald, Whittaker functions and Laguerre polynomials. More-over,They discussed some applications of extended beta function related to Statistics.

In 2017, Parmar et al.[20] introduced another interesting extension of the generalized beta function using modified bessel function.Further,making use of this extended Beta function they established other extensions of the extended Gauss hypergeometric and the confluent hypergeometric functions.They also investigated several properties of each of these extended functions, such as their various integral representations, Mellin transforms, derivatives, transformations, summation formulas, generating function and asymptotics.

In 2017,Pucheta [22] introduced a different extension of Gamma and Beta functions in terms of integral whose kernel contains the Mittag-Leffler function.Some basic properties were presented such as the integral representations, functional relations, Mellin transform of the new modified extended Beta function.

In 2017, Mubeen et al.[15] introduced a (p, q) -extended beta function by using the product of two confluent hypergeometric functions, which reduces to some known extended beta functions as well as the classical beta function. Then they investigated its several properties such as Mellin transform, integral representations, finite and infinite series representations. Further, they also established some roperties related to statistics.

In 2018, Al-Gonah and Mohammed[3] introduced and studied a new extension of extended Gamma and Beta functions in terms of integral whose kernel contains the generalized Mittag-Leffler function.They also obtained Some functional relations, integral representations and Mellin transform for these extended functions.

In 2018, Enes Ata[5] used Wright function to define new generalizations of gamma and beta functions.These functions were named as ψ-gamma and ψ-beta functions, respectively.Other than this he also gave ψ-generalization of Gauss and confluent hypergeometric functions and their integral representations,mellin transform and trasformation formulas.

The Mittag-Leffler function and its generalization appear in special functions as a solution of fractional order integral and differential equations.The Mittag-Leffler function was first defined and studied by the Swedish mathematician G. Mittag-Leffler(1903). It is a direct generalization of the exponential series.Later Wiman(1905), Prabhakar (1971) defined this function with some new parameter and discussed its different properties [14, 21, 31].

In 2007,Shukla and Prajapati[29] introduced a extension of Mittag-Leffler function. Its various properties including usual differentiation and integration, Laplace transforms,Euler (Beta) transforms, Mellin transforms, Whittaker transforms, generalised hypergeometric series form, Mellin–Barnes integral representation with their several special cases were obtained and its relationship with Laguerre polynomials, Fox H-function and Wright hypergeometric function have also been established.

In 2009, Tarique O. Salim [26] studied a generalized function of Mittag-Leffler type. Various results including differentiation formulas,Beta transform, Laplace transform, Mellin transform formulas and representation in terms of Wright function, Fox's H-function and generalized hypergeometric function have been established and proved.

In 2009,Srivastava and Tomovski[30] introduced generalized Mittag–Leffler functions and investigated a fractional calculus with an integral operator which contains this generalized Mittag–Leffler functions.

In 2012, Salim et al.[27] studied a new generalized function of Mittag-Leffler type and its various forms and properties including differentiation, transformation formulas, Mellin-Barnes integral representation and its relationship with Fox's H-function and Wright hypergeometric function. Further properties of generalized Mittag- Leffler function associated with fractional differential and integral operators were also considered.

In 2012, Kurulay and Bayram[13] obtained a number of useful relationships between the Mittag-Leffler functions and the Wright functions.

In 2013, Khan and Ahmed[12] have defined Mittag -Leffler function with more parameters and discussed various properties including usual differentiation and integration, Euler(Beta) transforms, Laplace transforms,Whittaker transforms, generalized hypergeometric series forms. Relationship with Wright hypergeometric function and Laguerre polynomials were also established.

In 2014, Ozarslan and Yılmaz[18] obtained the extended Mittag-Leffler functions by using the extended Beta functions, In addition,the Mellin transform of these functions was given in terms of generalized Wright hypergeometric functions.They also presented some relationships between these functions and the Laguerre polynomials and Whittaker functions.

The concept of extension of fractional operators attracts the attention of many researchers.
In 2010,M. Ali Özarslan and Emine Özergin[19] introduced the extended Riemann–Liouville fractional derivative operator with the help of Extended beta function defined by Chaudhary et al.[8].

In 2015, Baleanu et al.[5] defined the generalized beta function. They extended the definition of the fractional derivative operator of the Riemann-Liouville and discussed its properties . Moreover, they established some relations to extended special functions of two and three variables via generating functions.

In 2015,Agarwal et al.[1] presented an extension of the Riemann-Liouville fractional derivative operator and investigated its various useful and new properties and formulas, for example, integral representations, Mellin transforms, generating functions, and the extended fractional derivative formulas for some familiar functions.

In 2017, Agarwal et al.[2] introduced a class of new extended forms of the beta function, Gauss hypergeometric function and Appell and Lauricella hypergeometric functions by means of the modified Bessel function of the third kind. Some typical generating relations for these extended hypergeometric functions were obtained by defining the extension of the Riemann-Liouville fractional derivative operator.

In 2018,Rahman et al.[24] presented a new extension of the fractional derivative operator. Moreover, they established some results related to the newly defined modified fractional derivative operator such as Mellin transformand relations to extended hypergeometric and Appell's function via generating functions.

In 2018 Shadab et al. [28] introduced another form of the extended Beta function. and gave concept of a new and interesting Riemann–Liouville type fractional derivative operator. They have associated with the extended Appell hypergeometric functions of two variables and Lauricella hypergeometric function of three variables. With a view on analytic properties and application of new Riemann–Liouville type fractional derivative operator, they obtained new fractional derivative formulas for some familiar functions and for Mellin transformation formulas and established some new generating functions for an extended hypergeometric function using the new definition of fractional derivative operator.

In,2018,Bohner et al. [13] gave a further extension of the extended Riemann–Liouville fractional derivative operator by using an extended beta function defined by Parmar et. al.[20] by considering the Bessel functions in its kernel.They also gave some results related to the newly defined fractional operator, such as Mellin transform and relations to extended hypergeometric and Appell's function via generating functions.

In 2019,Rahman et al.[25] introduced an extended hypergeometric function,an extended confluent hypergeometric function, and an extension of the Appell function. They also, used the extended beta function to introduce an extended Riemann-Liouville type integral operator and investigated its associated formulas and generating relations.

In 2019,K.S.Nisar et al.[16] presented a new extension of the fractional derivative operator. Moreover, They have established some results related to the newly defined modified fractional derivative operator such as Mellin transform and relations to extended hypergeometric and Appell's function via generating functions.

3 Motivation

3.1 Some definitions and earlier work

3.1.1 The Gamma function

In the integer-order calculus the factorial plays an important role because it is one of the most fundamental combinatorial tools. The Gamma function has the same importance in the fractional-order calculus and it is basically given by integral

$$\Gamma(x) = \int_0^\infty e^{-t}t^{x-1}dt, \qquad Re(x) > 0. \tag{1}$$

The exponential provides the convergence of this integral when $t \to \infty$, the convergence at zero obviously occurs for all complex z from the right half of the complex plane $(Re(x) > 0)$. Other generalizations for values in the left half of the complex plane can be obtained in following way. If in (1) we substitute e^{-t} by the well-known limit

$$e^{-t} = \lim_{n\to\infty}\left(1 - \frac{t}{n}\right)^n$$

and then use n-times integration by parts, we obtain the following limit definition of the Gamma function

$$\Gamma(x) = \lim_{n\to\infty}\frac{n!n^x}{(x)_{n+1}} \tag{2}$$

where x is a complex number and $(a)_n$ is the Pochhammer symbol defined as

$$(a)_n = a(a+1)(a+2)...(a+n-1)$$

for $a \neq 0, (a)_0 = 1$.
Chaudhry and Zubair[9] have introduced the following extension of gamma function

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{(-t-\frac{p}{t})}dt, \qquad Re(x) > 0, Re(p) > 0. \tag{3}$$

3.1.2 The Beta Function

The Beta function is very important for the computation of the fractional derivatives of the power function. It is defined by the two-parameter integral

$$\mathcal{B}(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt, \qquad Re(x) > 0,\ Re(y) > 0. \tag{4}$$

The relation beteen Beta function and Gamma function is given by

$$\mathcal{B}(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma_k(x+y)}. \tag{5}$$

The beta function is extended to $\mathcal{B}_p(x,y)$ as follows [8]:

$$\mathcal{B}_p(x,y) = \mathcal{B}(x,y;p) = \int_0^1 t^{x-1}(1-t)^{y-1}{}_1e^{\frac{-p}{t(1-t)}}dt, \tag{6}$$

where $Re(x) > 0, Re(y) > 0, Re(p) > 0$. For $p = 0$ it gives the familiar beta function.
Choi[10] considered further extension (denoted by EEBF) of the extended Beta function$\mathcal{B}_p(x,y)$ by introducing an additional parameter q as follows:

$$\mathcal{B}(x,y;p,q) = \int_0^1 t^{x-1}(1-t)^{y-1}{}_1e^{\frac{-p}{t}-\frac{q}{(1-t)}}dt, \tag{7}$$

where $Re(x) > 0, Re(y) > 0, Re(p) > 0, Re(q) > 0$. It is easily seen that the cases $p = q$ and $p = 0 = q$ of (7) reduce to the extended Beta function (EBF) in (6) and the usual Beta function (4).
many more generalizations of beta function has been presented and investigated in [3, 7, 8, 15, 17, 22].

3.1.3 Hypergeometric function

The Hypergeometric functions with three parameters a, b, c, two parameters a, b in the numerator and one parameter c in the denominator, are defined by

$$ {}_2F_1(a, b, c; z) = \sum_{n=0}^{\infty} \frac{(a)_n \ (b)_n}{(c)_n} \frac{z^n}{n!}, $$

for all $a, b, c,$ and $c \neq 0, -1, -2, -3, ..., |z| < 1,$ where
$(a)_n = (a)(a + 1)(a + 2)...(a + n - 1); (a)_0 = 1.$

3.1.4 Mittag Leffler function

The exponential function e^z is very important in the theory of integer-order differential equations. We can write it in a form of series:

$$ e^z = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k + 1)}, \qquad (z \in C). \tag{8} $$

The generalizations of this function, so called functions of the Mittag-Leffler type, play an important role in the theory of fractional differential equations (FDEs). The Mittag-Leffler function $E_\sigma(z)$,

$$ E_\sigma(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\sigma \ n + 1)}, \qquad (z \in C; R(\sigma) > 0). \tag{9} $$

was defined and studied by Mittag -Leffler[14] in the year 1903. It is a generalization of the exponential series.since for $\sigma = 1$,we have the exponential function.Wiman[31] introduced a generalization of the Mittag-Leffler function.

$$ E_{\sigma,\eta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\sigma \ n + \eta)}, \qquad (z, \eta \in C; R(\sigma) > 0, R(\eta) > 0). \tag{10} $$

After that Prabhakar[21] introduced and investigated the following generalization of the function $E_{\sigma,\eta}(z)$by

$$ E^{\delta}_{\sigma,\eta}(z) = \sum_{n=0}^{\infty} \frac{(\delta)_n}{\Gamma(\sigma \ n + \eta)} \frac{z^n}{n!}, \qquad (z, \eta, \delta, \sigma \in C; R(\sigma) > 0, R(\delta) > 0, R(\eta) > 0). \tag{11} $$

where $(\lambda)_n$ denotes the Pocchammer symbol defined (for $\lambda \in C$) by:

$$ (\lambda)_n = \begin{cases} 1 & (n = 0) \\ \lambda(\lambda + 1)...(\lambda + n - 1) & (n \in N). \end{cases} \tag{12} $$

Shukla and Prajapati[29]introduced a generalized Mittag Leffler function $E^{\delta,c}_{\sigma,\eta}(z)$ by

$$E^{\delta,c}_{\sigma,\eta}(z) = \sum_{n=0}^{\infty} \frac{(\delta)_{nc}}{\Gamma(\sigma\, n+\eta)} \frac{z^n}{n!}, \tag{13}$$

$$(z, \eta, \delta, \sigma \in C; min\{R(\sigma), R(\eta), R(\delta)\} > 0; c \in (0,1) \cup N).$$

$\ddot{O}$zarslan and Yilmaz[18] introduced the following extended Mittag-Leffler function $E^{\delta,c}_{\sigma,\eta}(z;p)$ by

$$E^{\delta,c}_{\sigma,\eta}(z;p) = \sum_{n=0}^{\infty} \frac{\mathcal{B}_p(\delta+n, c-\delta)}{\mathcal{B}(\delta, c-\delta)} \frac{(c)_n}{\Gamma(\sigma\, n+\eta)} \frac{z^n}{n!}, \tag{14}$$

$$(z, \eta, \delta, \sigma \in C; R(c) > R(\delta) > 0, R(\sigma) > 0, R(\eta) > 0).$$

many more generalizations of Mittag-Leffler function were also investigated in [4, 12, 26, 27].

3.1.5 Wright Functions

In this section we briefly introduce the Wright function and mention some relations to functions of the Mittag-Leffler type. The general Wright function is defined by series:

$$\begin{aligned} {}_p\Psi_q(z) &= {}_p\Psi_q\left[\begin{matrix}(\lambda_i, \eta_i)_{1,p} \\ (\mu_i, \zeta_i)_{1,q}\end{matrix} ; z\right] \\ &= \sum_{n=0}^{\infty} \frac{\Gamma(\lambda_1+\eta_1 n)\cdots\Gamma(\lambda_p+\eta_p n)}{\Gamma(\mu_1+\zeta_1 n)\cdots\Gamma(\mu_q+\zeta_q n)} \frac{z^n}{n!}, \end{aligned} \tag{15}$$

where the coefficients $\eta_r (r = 1, ..., p)$ and $\zeta_s (s = 1, ..., q)$ are positive real numbers such that

$$1 + \sum_{s=1}^{q} \zeta_s - \sum_{r=1}^{p} \eta_r \geq 0.$$

The Relation between Wright functions and functions of the Mittag-Leffler type is as follows:

$$E_{\alpha,\beta}(z) = {}_1\Psi_1\left[\begin{matrix}(1,1)_{1,p} \\ (\beta_i, \alpha_i)_{1,q}\end{matrix} ; z\right]. \tag{16}$$

3.1.6 The Laplace Transform

The Laplace transform is a very useful tool for solving linear ODEs with constant coefficients since it converts linear differential equations to linear algebraic equations which can be solved easily. The final step, the inverse transform of the result, is usually the most complicated part of this approach. The Laplace transform of a function $f(t), (t > 0)$, is defined as

$$\mathcal{L}[f(t)] = F(s) = \int_0^{\infty} e^{-st} f(t) dt, \tag{17}$$

where s can be either real or complex.
The Laplace transform of $f(t)$ exists, if the integral converges for some value of s.

Correspondingly $f(t)$ is called the inverse Laplace transform of $F(s)$ which we denote as

$$f(t) = \mathcal{L}^{-1}\{F(s)\}.$$

3.1.7 The Mellin Transform

The Mellin transform of a function $f(t), (t > 0)$, is defined as

$$\mathcal{M}[f(t)](s) = F(s) = \int_0^\infty t^{s-1} f(t)dt, \tag{18}$$

where s can be either real or complex.
Correspondingly $f(t)$ is called the Mellin Laplace transform of $F(s)$ which we denote as

$$f(t) = \mathcal{M}^{-1}\{F(s)\} = \frac{1}{2\pi\iota}\int_{c-\iota\infty}^{c+\iota\infty} t^{-s} f(s)ds.$$

3.1.8 Riemann-Liouville Fractional Integral

Let v be a nonnegative number and f be a piecewise continuous on $J'(0,\infty)$and integrable on any finite subinterval of $J'[0,\infty]$. Then the Riemann-Liouville fractional integral of f of order v is defined by

$$D_x^{-v} f(x) = \frac{1}{\Gamma v}\int_0^x (x-t)^{v-1} f(t)dt, \qquad Re(v) > 0. \tag{19}$$

3.1.9 Riemann-Liouville Fractional Derivative

The Riemann -Liouville fractional derivative operator of order v is defined by

$$D_z^\nu (f(z) = \frac{1}{\Gamma(-\nu)}\int_0^z (z-t)^{\nu-1} f(t)dt, \qquad Re(\nu) < 0. \tag{20}$$

where the integration path is a line from 0 to z in the complex t-plane.
For the case $k-1 < Re(\nu) < k$(k=1,2,3,...) is defined by

$$\begin{aligned} D_z^\nu f(z) &= \frac{d^k}{dx^k}[D_z^{\nu-k} f(z)] \\ &= \frac{d^k}{dx^k}\Big\{\frac{1}{\Gamma(-\nu+k)}\int_0^z (z-t)^{-\nu+k-1} f(t)dt\Big\}, \qquad Re(\nu) < 0. \end{aligned} \tag{21}$$

The extended R-L fractional derivative of order μ is defined in [1]by

$$D_z^{\nu,p}(f(z) = \frac{1}{\Gamma(-\nu)}\int_0^z f(t)(z-t)^{-\nu-1} exp\Big(\frac{-pz^2}{t(z-t)}\Big)dt, \qquad Re(\nu) < 0,\ Re(p) > 0. \tag{22}$$

For the case $k-1 < Re(\nu) < k$ (k=1,2,3,...) it follows

$$\begin{aligned} D_z^{\nu,p}(f(z)) &= \frac{d^k}{dx^k}[D_z^{\nu-k,p} f(z)] \\ &= \frac{d^k}{dx^k}\Big\{\frac{1}{\Gamma(-\nu+k)}\int_0^z f(t)(z-t)^{\nu+k-1} exp\Big(\frac{-pz^2}{t(z-t)}dt\Big)\Big\}, \end{aligned}$$

$$(Re(\nu) < 0, ; Re(p) > 0).$$

many reasearchers[1, 2, 6, 13, 16, 24] investigated the various extensions and generalizations of fractional derivative operators.

3.1.10 Caputo-fractional derivative

The Caputo-fractional derivative operator of order$v$ is defined by

$$D_z^{\nu}(f(z) = \frac{1}{\Gamma(m-\nu)} \int_0^z (z-t)^{m-\nu-1} \frac{d^m}{dx^m} f(t)dt, \tag{23}$$

where $m-1 < Re(\nu) < m$.

3.2 Summary of the proposed work

3.2.1 On a ψ-generalized Riemann -Liouville fractional derivative operator and its applications

In 2018 Enes Ata[5] has introduced the following generalizations of gamma and beta functions as

$$^{\psi}\Gamma_p^{(\alpha,\beta)}(x) = \int_0^{\infty} t^{x-1} {}_1\Psi_1\Big(\alpha, \beta; -t - \frac{p}{t}\Big)dt, \tag{24}$$

where $Re(x) > 0, Re(\alpha) > 0, Re(\beta) > 1, p > 0, \alpha, \beta \in C$ and ${}_1\Psi_1(\alpha, \beta, z)$ is the Wright function defined by

$${}_1\Psi_1(\alpha, \beta, z) = \sum_{n=0}^{\infty} \frac{1}{\Gamma(\alpha n + \beta)} \frac{z^n}{n!}.$$

and

$$^{\psi}\mathcal{B}_p^{(\alpha,\beta)}(x, y) = \int_0^{\infty} t^{x-1}(1-t)^{y-1} {}_1\Psi_1\Big(\alpha, \beta; \frac{-p}{t(1-t)}\Big)dt, \tag{25}$$

where $Re(x) > 0, Re(x) > 0, Re(\alpha) > 0, Re(\beta) > 1, Re(p) > 0$. this new generalizations of gamma and beta functions was called as ψ-gamma and ψ-beta functions,respectively.It was observed that

$$^{\psi}\Gamma_p^{(0,2)}(x) = \Gamma_p(x)$$

$$^{\psi}\Gamma_0^{(0,2)}(x) = \Gamma(x)$$

and

$$^{\psi}\mathcal{B}_p^{(0,2)}(x, y) = \mathcal{B}_p(x, y)$$

$$^{\psi}\mathcal{B}_0^{(0,2)}(x, y) = \mathcal{B}(x, y).$$

He also gave ψ-generalization of Gauss and Confluent hypergeometric function as

$$^{\psi}\mathcal{F}_p^{(\alpha,\beta)}(a, b; c; z) = \sum_{n=0}^{\infty} (a)_n \frac{^{\psi}\mathcal{B}_p^{(\alpha,\beta)}(b+n, c-b)}{\mathcal{B}(b, c-b)} \frac{z^n}{n!} \tag{26}$$

and

$$^{\psi}\Phi_p^{(\alpha,\beta)}(b; c; z) = \sum_{n=0}^{\infty} \frac{^{\psi}\mathcal{B}_p^{(\alpha,\beta)}(b+n, c-b)}{\mathcal{B}(b, c-b)} \frac{z^n}{n!}. \tag{27}$$

Ata called ${}^{\psi}\mathcal{F}_p^{(\alpha,\beta)}(a,b;c;z)$ as ψ-Gauss hypergeometric function and ${}^{\psi}\Phi_p^{(\alpha,\beta)}(b;c;z)$ as ψ-Confluent hypergeometric function. We observe that

$$
\begin{aligned}
{}^{\psi}\mathcal{F}_p^{(0,2)}(a,b;c;z) &= \mathcal{F}_p(a,b;c;z)\\
{}^{\psi}\mathcal{F}_0^{(0,2)}(a,b;c;z) &= {}_2\mathcal{F}_1(a,b;c;z)\\
{}^{\psi}\Phi_p^{(0,2)}(b;c;z) &= \Phi_p(b;c;z)\\
{}^{\psi}\Phi_p^{(0,2)}(b;c;z) &= \Phi(b;c;z)
\end{aligned}
$$

where$\mathcal{F}_p(a,b;c;z)$ and $\Phi_p(b;c;z)$ are generalized functions.

Ata has also obtained integral representation of ψ-Gauss hypergeometric function as

$$
{}^{\psi}\mathcal{F}_p^{(\alpha,\beta)}(a,b;c;z) = \frac{1}{\mathcal{B}(b,c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}{}_1\Psi_1\Big(\alpha,\beta;\frac{-p}{t(1-t)}\Big)(1-zt)^{-a}dt, \quad (28)
$$

where $Re(p)>0, Re(c)>Re(b)>0, |arg(1-z)|<\pi$.
and integral representation of ψ-Confluent hypergeometric function as

$$
{}^{\psi}\Phi_p^{(\alpha,\beta)}(b;c;z) = \frac{1}{\mathcal{B}(b,c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}{}_1\Psi_1\Big(\alpha,\beta;\frac{-p}{t(1-t)}\Big)dt, \quad (29)
$$

where $Re(p)>0, Re(c)>Re(b)>0, |arg(1-z)|<\pi$.
Further, the author also established differentiation formulas, mellin transform and transformation formulas for these ψ-generalized gauss hypergeometric function and confluent hypergeometric function.

Now, we propose to define ψ-generalization of Riemann Liouville fractional derivative operator

$$
{}^{\psi}D_{z,p}^{\nu,\alpha,\beta}(f(z)) = \frac{1}{\Gamma(-\nu)}\int_0^z f(t)(z-t)^{-\nu-1}{}_1\Psi_1\Big(\alpha,\beta,\frac{-pz^2}{t(z-t)}\Big)dt, \quad (30)
$$

where $Re(\nu)<0,\ Re(p)>0, Re(\alpha)>0, Re(\beta)>0$.
For the case $k-1<Re(\nu)<k$ (k=1,2,3,...) it follows

$$
\begin{aligned}
{}^{\psi}D_{z,p}^{\nu,\alpha,\beta}(f(z)) &= \frac{d^k}{dx^k}[D_{z,p}^{\nu-k,\alpha,\beta}f(z)]\\
&= \frac{d^k}{dx^k}\Big\{\frac{1}{\Gamma(-\nu+k)}\int_0^z f(t)(z-t)^{\nu+k-1}{}_1\Psi_1\Big(\alpha,\beta,\frac{-pz^2}{t(z-t)}\Big)dt\Big\},
\end{aligned}
$$

where the integration path is a line from 0 to z in the complex t-plane.
For $p=0$ and $\alpha=0$ and β=2 we obtain the original Riemann-Liouville fractional derivative operator.

we may also establish integral representations of ψ-generalized hypergeometric functions. Moreover, we may obtain fractional derivative formulas for some familiar functions and some useful properties.

3.2.2 On a ψ-generalized Mittag Leffler function and its properties

In 2018 Enes Ata[5] introduced the following generalizations of gamma and beta functions as

$$ {}^{\psi}\mathcal{B}_{p}^{(\alpha,\beta)}(x,y) = \int_{0}^{\infty} t^{x-1}(1-t)^{y-1}{}_{1}\Psi_{1}\Big(\alpha,\beta;\frac{-p}{t(1-t)}\Big)dt, \tag{31} $$

$$ (Re(x)>0, Re(\alpha)>0, Re(\beta)>1, Re(p)>0). $$

For $p=0$ and $\alpha=0$ and β=2 these give the original beta function.
We propose to define ψ-generalization of the extended Mittag-Leffler function as follows:

$$ {}^{\psi}E_{\mu,\sigma,p}^{\gamma,c,\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{{}^{\psi}\mathcal{B}_{p}^{(\alpha,\beta)}(\gamma+n, c-\gamma)}{\mathcal{B}(\gamma, c-\gamma)} \frac{(c)_n}{\Gamma(\mu\, n+\sigma)} \frac{z^n}{n!}, \tag{32} $$

$$ (z,\eta,\delta,\sigma \in C; R(c)>R(\gamma)>0, R(\sigma)>0, R(\mu)>0, p\geq 0) $$

where ${}^{\psi}\mathcal{B}_{p}^{(\alpha,\beta)}(x,y)$ is ψ-generalized beta function defined by(31).
For $p=0$ and $\alpha=0$ and β=2 (32) gives (11).
For $p\neq 0$ and $\alpha=0$ and β=2 (32) gives (14).
we also propose to establish integral representations of this ψ-generalized Mittag Leffler function. Also, we may obtain some useful properties for example, Mellin transforms, recurrence relations and derivative formulas for this new ψ-generalized mittag-leffler function.

Moreover, we also propose to study various types of the fractional differential equations, fractional integral inequalities, integral equation and integral trasform with their application in statistics and applied mathematics.

4 Objective

- To establish some new generalisation of known special functions by using extended gamma and beta functions.
- To present more generalization of fractional differential and integral operators.
- To investigate the integral equations and integral transforms using extended gamma and beta functions.
- To obtain certain new fractional integral inequalities involving generalized hypergeometric functions and fractional differ-integral operators.
- To study the fractional calculus and special functions with probability distributions and fluid dynamics as an application.
- To study concept of fractional differential equations.

5 Plan of work and Methodology

5.1 Plan of Work

The Research plan will be carried out as follows:

- We will study the various special functions and fractional derivative and integral operators.
- We propose to develop some new generalizations of the gamma,beta and hypergeometric functions and may apply in different fields like physics,fluid dynamics etc.
- We propose to establish new forms of fractional derivative and integral operator with some properties.
- We propose to establish some applications of extended gamma,beta and hypergeometric functions and fractional differ-integral operators.
- By using concept of fractional differential equations we propose to solve some problems by matlab/scilab, mathematica.
- We propose to develop new fractional integral inequalities like chebyshev, Grüsstype and manymore.

5.2 Methodology

- **Procedure**
  (i) To identify the problem
  (ii) Mathematical formulation of the problem
  (iii) Solution using suitable method
  (iv)To analyse the numerical result and conclusion

- **Solution Method**
  (i) Using the convergence method like comparison test, cauchy's root test etc
  (ii) series solution and expansions of function
  (iii) existence conditions
  (iv)Numerical methods

- **Softwares**
  (i) Matlab/scilab
  (ii)Mathematica
  (iii)Latex

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